Math Challenge 5. Evaluate $$\sqrt[\ldots]{7 - \frac{1}{7 - \frac{1}{\ldots}}}.$$ 

Express your answer in the form $$\frac{a + b\sqrt{c}}{d}$$ where $$a, b, c, d \in \mathbb{Z}$$.

**Solution.** The nested fraction under the big radical notation is the limit of the sequence that can be recursively defined by

$$x_1 = 7, \quad x_{n+1} = 7 - \frac{1}{x_n} \quad \text{for} \quad n \geq 1.$$ 

By using mathematical induction, one can show that the sequence $$(x_n)$$ is a monotone decreasing sequence bounded below by 6. Thus, $$(x_n)$$ is convergent. Let $$x = \lim_{n \to \infty} x_n$$. Then by the recurrence relation above, $$x$$ satisfies the equation

$$x = 7 - \frac{1}{x} \iff x^2 - 7x + 1 = 0.$$ 

Then using the quadratic formula, we obtain $$x = \frac{7 \pm \sqrt{5}}{2}$$. Because the sequence $$(x_n)$$ is bounded below by 6, we must have $$x \geq 6$$, i.e., $$x = \frac{7 + 3\sqrt{5}}{2}$$.

The solution to the problem is then

$$\sqrt{x} = \sqrt{\frac{7 + 3\sqrt{5}}{2}} = \frac{\sqrt{7 + 3\sqrt{5}}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{14 + 6\sqrt{5}}}{2}$$

$$= \frac{\sqrt{9 + 6\sqrt{5} + 5}}{2} = \frac{\sqrt{(3 + \sqrt{5})^2}}{2} = \frac{3 + \sqrt{5}}{2}$$

as requested.

A general formula provided by Dr. Bangteng Xu:

$$\sqrt[\ldots]{\frac{n - \frac{1}{n - \frac{1}{\ldots}}}{n - \frac{1}{\ldots}}} = \frac{\sqrt{n+2} + \sqrt{n-2}}{2}$$