Math Challenge 2. Find the exact value of \( a > 0 \) that maximizes the area between the graph of

\[ f(x) = x^a(1 - x^a) \]

and the \( x \)-axis from \( x = 0 \) to \( x = 1 \). Use some test to verify that you actually have the maximum area.

Solution. The area under the graph of \( f \) from \( x = 0 \) to 1 is

\[
A(a) = \int_0^1 f(x) \, dx = \int_0^1 (x^a - x^{2a}) \, dx = \left[ \frac{x^{a+1}}{a+1} - \frac{x^{2a+1}}{2a+1} \right]_0^1 = \frac{1}{a+1} - \frac{1}{2a+1}.
\]

Here we find the value of \( a \) that maximizes \( A(a) \) over the interval \((0, \infty)\).

We compute \( \frac{dA}{da} = -\frac{1}{(a+1)^2} + \frac{2}{(2a+1)^2} \) and then solve \( \frac{dA}{da} = 0 \) for \( a \).

\[
\frac{dA}{da} = 0 \iff \frac{2}{(2a+1)^2} = \frac{1}{(a+1)^2} \\
\iff 2(a+1)^2 = (2a+1)^2 \\
\iff 2a^2 + 4a + 2 = 4a^2 + 4a + 1 \\
\iff 1 = 2a^2 \iff a = \frac{1}{\sqrt{2}} \quad \text{(because } a > 0\text{)}.
\]

Note that \( A(a) \) is continuous and differentiable on \((0, \infty)\) and that \( A'(a) > 0 \) when \( 0 < a < 1/\sqrt{2} \) and \( A'(a) < 0 \) when \( a > 1/\sqrt{2} \). Thus, by the Increasing/Decreasing Test, \( A \) is increasing on \((0, 1/\sqrt{2})\) and decreasing on \((1/\sqrt{2}, \infty)\).

Therefore, \( A(a) \) attains its absolute maximum when \( a = 1/\sqrt{2} \).