The Exchange Condition for Association Schemes

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Abstract. Let X be a set, and let S be a partition of $X \times X$ such that $1_X \in S$ and, for each element s in S, $s^* := \{(y,z) \mid (z,y) \in s\} \in S$. The set S is called a *scheme on* X if, for any three elements p, q, and r in S, there exists a cardinal number a_{pqr} such that, for any two elements y and z in X with $(y,z) \in r$, there exist exactly a_{pqr} elements x in X with $(y,x) \in p$ and $(x,z) \in q$.

Let G be a group, and let H be a subgroup of G. For each element g in G, we define g^H to be the set of all pairs (eH, egH). It is easy to see (and well-known) that $\{g^H \mid g \in G\}$ is a scheme on $\{gH \mid g \in G\}$. A scheme is called *schurian* if it arises from a group in the above-described way. Schurian schemes with $H = \{1\}$ are called *thin*. Thin schemes can be identified with groups.

Let X be a set, and let S be a scheme on X. For each nonempty subset R of S, we define R^* to be the set of all elements r^* with $r \in R$. For any two nonempty subsets p and q of S, we define pq to be the set of all elements s in S satisfying $a_{pqs} \neq 0$. A nonempty subset R of S is called *closed* if, for any two elements p and q in R, $p^*q \subseteq R$.

An element s of the scheme S is called an *involution* if $\{1, s\}$ is closed. In my talk, I will define the exchange condition for involutions of schemes in such a way that its restriction to thin schemes coincides with the group theoretic exchange condition which distinguishes the Coxeter groups among the groups generated by involutions.

The main purpose of my talk will be a scheme theoretic description of buildings and twinned buildings (in the sense of Jacques Tits) with the help of the above-mentioned exchange condition.