Monochromatic pieces in edge colorings of complete graphs

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A first exercise in Graph Theory (in fact it is indeed the first in Bollobás: Modern Graph Theory) says that either a graph or its complement is connected. One can also state this observation as follows: in every 2-coloring of the edges of a complete graph K_n there is a monochromatic connected subgraph on n vertices. Or - perhaps this is the most attractive formulation - one can say that there is a monochromatic spanning tree in every 2-coloring of the edges of a complete graph. What happens if we color the edges of K_n with more than two, say t colors? The answer is that there is a monochromatic connected subgraph with at least $\frac{n}{t-1}$ vertices. There is equality here if $(t-1)^2$ is a divisor of n and $t-1$ is a power of prime (find a 4-coloring of the edges of K_{9k} without monochromatic subtree of order $3k+1$). I will show a new easy proof (joint work with Gábor Sárközy), of a stronger result: if the edges of K_n are colored with $t \geq 3$ colors, there is a monochromatic double star with at least $\frac{n}{t-1}$ vertices (what happens for $t=2$?). (A double star is a tree obtained by joining the centers of two disjoint stars.) Several natural variations and generalizations of this problem have been looked at, I collected some below.

1. Can one say something about a monochromatic spanning tree present in every 2-coloring of K_n? (Can it be a certain type of tree?)

2. Instead of edges of K_n, color the edges of K_n^3 (triples of an n-element set) with t colors. How large monochromatic connected piece can be guaranteed? Here the answer is known for $t = 1, 2, 3, 4, 7$ but not for $t = 5, 6$. (What is the order of the largest monochromatic connected piece for $t = 3$? What if $t = 7$?)

3. The t-colorings of K_n are replaced with local t-colorings: one can color with arbitrary number of colors, but the edges incident to any given vertex receive at most t colors. (In every local 2-coloring of K_n there is a monochromatic tree with at least ? vertices. What happens for local 3-colorings?)

4. The 2-colorings of K_n are replaced by Gallai-colorings: one can color with arbitrary number of colors but no triangle can be colored with three different colors. (Exercise: in every Gallai coloring of the edges of a complete graph, there is a monochromatic spanning tree.)

5. Instead of connected pieces, one can look for other type of (monochromatic) pieces: pieces without isolated points, matchings, connected matchings, pieces of high connectivity, pieces of small diameter.